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Optimization of conduits' shape in micro heat exchangers

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Abstract

An approximate theory is derived to compute the thermal resistances of flat plate micro heat exchangers whose surfaces are heated with uniform flux. It is demonstrated that the thermal resistance can be minimized by proper selection of uniform conduit geometry. Further reductions in the maximum heated surface temperature and in the heated surface temperature gradients can be achieved by varying the conduit's cross-sectional dimensions as a function of the axial coordinate. This paper illustrates that a conduit's shape can be customized so as to achieve desired objectives.

Nomenclature

a conduit width [m]
b conduit depth [m]
C_p specific heat at constant pressure [W(kg K)$^{-1}$]
D_H hydraulic diameter
h heat transfer coefficient [W(m^2 K)$^{-1}$]
k thermal conductivity [W(m K)$^{-1}$]
L conduit length
m nondimensional mass flow rate
$\dot{m}_0$ mass flow rate scale [kg s^{-1}] [$\dot{m}_0 = (b^4 \Delta p / \nu L)$]
q'' uniform heat flux at the heated surface [W m^{-2}]
R nondimensional thermal resistance
T temperature [K]
w conduit and fin width [m]
x axial coordinate.

Greek symbols

α nondimensional conduit width
Δp pressure drop [Pa]
θ nondimensional temperature
μ viscosity [kg(m s)$^{-1}$]
ν kinematic viscosity
τ average wall shear stress [Pa].

Subscripts

b bulk
cal calorimetric
c-s cross-sectional
f fluid
s solid.

Superscript

* dimensional quantities.

Nondimensional groups

Nu Nusselt number [$Nu = (hD_H/k_f)$]
Po Poiseuille number
η $q''b/k_fT_0$
χ $k_fL/C_p\dot{m}_0$.

1. Introduction

Micro heat exchangers are an efficient means for removing high heat fluxes with relatively small temperature gradients (i.e. [1–3]). In the design of micro heat exchangers, one strives to minimize either the maximum temperature or the temperature gradients of the heated surface. Temperature gradients are undesirable since, among other things, they may cause non-uniform thermal expansion, thermal stresses, and mechanical fatigue, particularly at the interface between two dissimilar materials. Thermal gradients and stresses may also adversely affect semiconducting properties.

To date, most studies of micro heat exchangers have focused on conduits with a uniform cross-section in the flow direction. A great amount of attention has been given to optimizing the cross-sectional dimensions so as to minimize the maximum temperature of the heated surface [4–7]. To see whether optimal cross-sectional dimensions exist when the pressure drop across the channel is given, one can follow the simple arguments given

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below. The heat transfer process can be characterized approximately by two thermal resistances. One thermal resistance, the calorimetric resistance, R_{cal}, is proportional to the difference between the conduit's exit and the inlet's bulk temperatures. R_{cal} is inversely proportional to the mass flow rate. When the pressure drop across the conduit's length is given, R_{cal} will decrease as the conduit's hydraulic diameter increases. The second resistance, the cross-sectional resistance $R_{c\text{-}s}(x)$, is proportional to the difference between the heated surface's temperature and the bulk temperature. In general, the cross-sectional resistance is a function of the axial coordinate, x. When the flow is fully developed and the conduit is uniform $R_{c\text{-}s}$ is independent of x. Since often the thermal conductivity of the solid substrate is much larger than that of the fluid, $R_{c\text{-}s}$ will decrease as the conduit's hydraulic diameter decreases. Since $R_{c\text{-}s}$ and R_{cal} vary in opposite ways as the conduit's cross-sectional dimensions vary, one would expect that there are optimal conduit dimensions that minimize the total resistance, $R_{cal}+R_{c\text{-}s}$.

In the case of a non-uniform conduit, one can do even better. The heated surface temperature increases as a function of the x-coordinate. In order to reduce the maximum, downstream surface temperature, it is sufficient to maintain a low cross-sectional resistance in the vicinity of the conduit's exit. Close to the conduit's entrance, one can afford to have a relatively large cross-sectional thermal resistance. In other words, one can maintain a larger hydraulic diameter next to the conduit's entrance which, in turn, would lead to a reduction in R_{cal}. The objective of this note is to illustrate that a micro heat exchanger with a variable cross-section can provide better performance that a micro heat exchanger with a uniform cross-section. Since typically micro heat exchangers are fabricated utilizing photolithographic techniques, fabrication of micro heat exchangers with non-uniform cross-sections is feasible.

2. Mathematical model

The objective of this note is to demonstrate that by proper design of the conduit's geometry, one can obtain a better performance than is feasible with uniform cross-section conduits. For simplicity's sake, rigor is sacrificed.

The micro-heat exchanger is fabricated in silicon or some other suitable material. One surface of the heat exchanger is heated with a uniform heat flux ($q^{*\prime\prime}$). The opposite surface is insulated. The heat exchanger is equipped with many micro-conduits (as many as a few hundred). Each conduit has a rectangular cross-section of width $a^*(x)$ and depth b^*. The presence and absence of the superscript star (*) denote, respectively, dimensional and nondimensional quantities. A cross-section of a single conduit enclosed between two surfaces representing symmetry planes is depicted in Fig. 1. The figure also depicts the relevant dimensions.

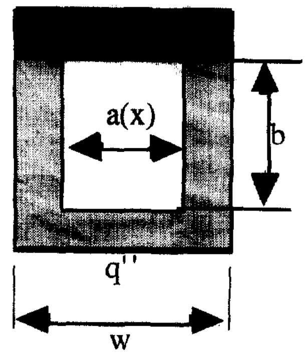

Fig. 1. A schematic description of the micro heat exchanger's cross-section (not drawn to scale).

The thermal conductivity of the substrate (k_s^*) is typically much higher than that of the fluid (k_f^*). For example, when the substrate is made out of silicon and the fluid is water at room temperature, $k_s^*/k_f^* \sim 243$. When the fluid is nitrogen at ~ 77 K, $k_s^*/k_f^* \sim 5000$. For simplicity, it is assumed that at each cross-section, the solid temperature is uniform and that axial conduction, both in the solid and fluid, can be neglected. Although these assumptions are not strictly correct, they will still allow us to obtain qualitatively correct results. For example, when Yin and Bau [6] used a similar assumption to derive an approximate formula for the optimal dimensions of a uniform cross-section conduit. The formula they derived was in good agreement with the results of conjugate, three-dimensional calculations in which axial conduction both in the liquid and the solid was included. Moreover, in all cases studied here, the use of non-uniform cross-sections leads to reductions in the axial temperature gradients which, in turn, further reduce the importance of axial conduction.

Yin and Bau [6] showed that it is desirable to make the conduit as deep as possible and that the optimal fin's thickness (w-a, see Fig. 1) is typically much smaller than what would be necessary to assure the fin's structural integrity during fabrication and operation. Hence, it is assumed here that the conduit's depth, b^*, and the minimal fin's thickness, $w_f^* = \min_x(w^*-a^*)$, are determined by structural considerations.

Next, the various variables are nondimensionalized. The conduit's depth, b^*, is the length scale; $q^{*\prime\prime}b^*/k_f^*$ is the temperature scale; and $\dot{m}_0^* = (b^{*4}\Delta p^*/v^*L^*)$ is the mass flow-rate scale. In the above Δp^* is the pressure drop across the length of the conduit, L^*; and v^* is the kinematic viscosity. The nondimensional channel width, $\alpha(x) = a^*(x)/b^*$, is a function of the nondimensional, axial coordinate x, where $0 \leqslant x \leqslant 1$ is normalized with the channel length, L^*. The nondimensional fin thickness, $w_f = w_f^*/b^*$.

The flow is assumed to be locally fully developed and incompressible. In other words, the friction factor for fully developed flow in a rectangular cross-section is being used. Prior experimental work [8] on compressible gas flow in microchannels suggests that, when the changes in the velocity profile are gradual, such an approximation introduces a relatively small error. The average wall shear stress (τ_w^*) is given by $\tau_w^* = (1/8)(\mu^* u^*/D_H^*)Po(\alpha)$, where μ^* is the shear viscosity: u^* is the cross-sectionally averaged velocity; $D_H^* = 2\alpha b^*/(1+\alpha)$ is the hydraulic diameter; and $Po(\alpha)$ is the Poiseuille number. The Poiseuille number can be correlated as a function of the aspect ratio, α [9].

$$Po(\alpha) = 96(1 - 1.3553\alpha + 1.9467\alpha^2 - 1.7012\alpha^3 + 0.9564\alpha^4 - 0.2537\alpha^5). \quad (1)$$

Mass conservation and a force balance yield the expression for the nondimensional mass flow rate,

$$m = \left(\frac{1}{8}\int_0^1 Po(\alpha)\frac{(1+\alpha)^2}{\alpha^3}\,\mathrm{d}x\right)^{-1}. \quad (2)$$

Witness that $\alpha = \alpha(x)$. In the derivation of equation (2), inertial effects were neglected. When the rate of change of the channel width as a function of x is small, acceleration contributes very little to the pressure loss.

To describe the heat interaction between the fluid and the conduit's walls, the data given in Shah and London [9] for the Nusselt number for a rectangular conduit with uniform temperature on three surfaces and an insulating condition on the fourth surface was correlated as:

$$Nu(\alpha) = 8.235(1 - 1.883\alpha + 3.767\alpha^2 - 5.814\alpha^3 + 5.361\alpha^4 - 2\alpha^5) \quad (0 < \alpha < 1). \quad (3)$$

The nondimensional temperature is $\theta(x) = (q^{*\prime\prime} b^*/k_f^*)^{-1}(T^*(x) - T_b^*(0))$, where $T_b^*(0)$ is the bulk temperature at the conduit's inlet. The cross-sectional resistance, $R_{c\text{-}s}$, is proportional to the temperature difference between the solid and the fluid's bulk temperature at any location, x.

$$R_{c\text{-}s} = \theta_s(x) - \theta_b(x) = \frac{2\alpha(\alpha_{max} + w_f)}{Nu(1+\alpha)(2+\alpha)} \quad (4)$$

where $\alpha_{max} = \max_{0 \leqslant x \leqslant 1}(\alpha(x))$ is the maximum channel width and w_f is the thickness of the solid fin as dictated by structural and fabrication considerations. Typically, but not necessarily, α_{max} would occur at the conduit's entrance. The calorimetric resistance is proportional to the difference between the bulk temeperature at location x downstream and the inlet bulk temperature.

$$R_{cal} = \theta_b(x) = \chi\frac{(\alpha_{max} + w_f)}{m}x \quad (5)$$

where $\chi = (k_f^* L^*/C_p^* \dot{m}_0^*)$ and C_p^* is the fluid's specific heat at a constant pressure. The solid's temperature distribution is given by:

$$\theta_s(x) = (\alpha_{max} + w_f)\left(\frac{2\alpha}{Nu(1+\alpha)(2+\alpha)} + \frac{\chi}{m}x\right). \quad (6)$$

3. The optimization problem

The design problem involves the identification of the optimal conduit geometry, $\alpha(x)$, needed to achieve the desired objectives. In other words, one wishes to determine α as a function of x. There are a number of possible objectives. For example,

$$J(C_1, C_2) = \min_{0 \leqslant x \leqslant 1}\left(C_1 \max_{0 \leqslant x \leqslant 1}(\theta_s(x)) + C_2\int_0^1\left(\frac{\mathrm{d}\theta_s(x)}{\mathrm{d}x}\right)^2\mathrm{d}x\right). \quad (7)$$

The functional $J(C_1, 0)$ represents an optimization problem in which one wishes to minimize the maximum surface temperature. This is not a proper variational problem, and it does not have a solution in the space of square integrable functions. This shortcoming is attributable to the fact that the model neglects axial temperature gradients. The optimal $\alpha(x)$ would consist of a very wide channel along most of the conduit's length and an abrupt contraction to a very short pinhole at the conduit's exit. Such a conduit geometry would, however, mandate at the point of contraction very high axial temperature gradients which would render the model invalid. In order to be able to demonstrate that, by varying α as a function of x, one can reduce the maximum temperature, the search for a minimum is restricted to the space of truncated polynomial functions, i.e.,

$$\alpha(x) = \sum_{i=1}^{N} \alpha_i x^i. \quad (8)$$

In other words, one wishes to determine the coefficients $\alpha_1, \ldots, \alpha_N$ that minimize $J(C_1, 0)$. The case of the uniform conduit corresponds to $\alpha_0 \neq 0$ and $\alpha_i = 0$ when $i > 0$.

The functional $J(0, C_2)$ corresponds to an optimization problem in which one wishes to minimize the temperature gradients. To pose this problem well, one needs to supplement equation (7) with appropriate boundary conditions. In principle, one should be able to derive Euler–Lagrange equations for the determination of the function $\alpha(x)$. The resulting Euler–Lagrange equations are, however, very complicated. Instead, I will again assume that $\alpha(x)$ has a polynomial form.

The functional $J(C_1, C_2)$ corresponds to a minimization problem in which one wishes to minimize both the maximum temperature and the temperature gradients. The relative importance that one assigns to each of the terms is dictated by the relative magnitude of the coefficients, C_i.

4. Results and discussion

First, a uniform conduit heat exchanger is discussed. For illustration purposes, consider water flowing in a conduit of depth 1500 μm, 1 cm in length, with a pressure drop of 20 kPa. The nondimensional fin thickness $w_f = 1/30$ and $\chi = 1.695 \cdot 10^{-6}$. The thermophysical properties of the water were taken at room temperature. As the conduit's width (α) decreases, the cross-sectional area available to the fluid decreases. Since the pressure drop is fixed, the mass flow rate decreases and the calorimetric resistance increases. At the same time, as α decreases, the cross-sectional resistance decreases. The cross-sectional, the calorimetric, and the total resistances are depicted in Fig. 2. Clearly, there is an optimal conduit width for which the total resistance is minimized.

The optimal conduit width is a function of χ. Figure 3 depicts the optimal width as a function of χ. The results of the optimization are depicted as solid circles. The data is correlated using least squares in the form:

$$\alpha_{\text{optimal}} = 1.7378(\chi)^{0.2455}. \tag{9}$$

The correlation is depicted as a solid line in Fig. 4. In the range $10^{-7} < \chi < 10^{-2.5}$, the correlation (9) is accurate within better than 3%.

Next, it is demonstrated that conduits with non-uniform widths can outperform the ones with optimal uniform widths. In contrast to the uniform-width conduit, the maximum surface temperature of the non-uniform conduit does not necessarily occur at the conduit's exit. Thus, it is necessary to minimize the maximum temperature in the interval $0 \leqslant x \leqslant 1$. In order to restrict the search for $\max(\theta)$ to the interval $x \in [0, 1]$, the penalty function,

$$\text{penalty}(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 1 \\ Kx^2(1-x)^2 & \text{otherwise} \end{cases} \tag{10}$$

is subtracted from θ. In the above, K is a very large number (i.e., $K = 10^{10}$). $\text{Max}(\theta)$ and $\min(\max(\theta))$ were computed using gradient descent techniques. All the computations were carried out in Mathematica [10].

This investigation is restricted to conduits having a shape that can be described by a quadratic polynomial, i.e., $N = 2$ in equation (8). The use of quadratic polynomials also allowed the evaluation of the integral (2) in a closed form. Since this expression is quite lengthy, it is not reproduced here. For illustration purposes, equation (7) is solved for $\alpha = \alpha_0 + \alpha_2 x^2$, $\chi = 1.695 \cdot 10^{-6}$, and $C_2 = 0$. The optimal values were $\alpha_0 = 0.78$ and $\alpha_2 = -0.024$. The surface temperature as a function of

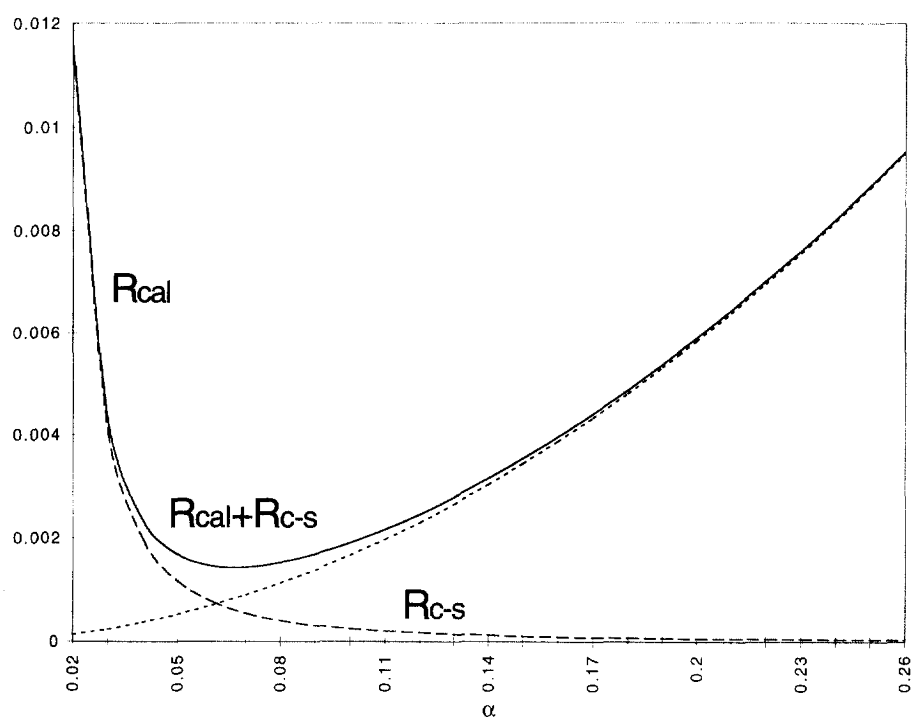

Fig. 2. The calorimetric, cross-sectional, and total resistances are depicted as a function of the non-dimensional width, α. $\chi = 1.695 \cdot 10^{-6}$.

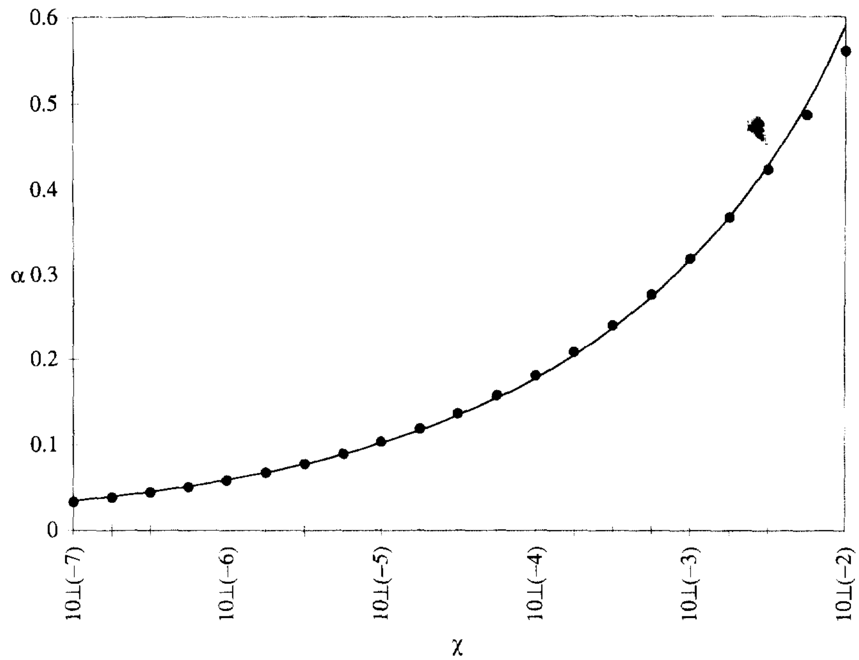

Fig. 3. The optimal nondimensional uniform conduit width (α) is depicted as a function of χ.

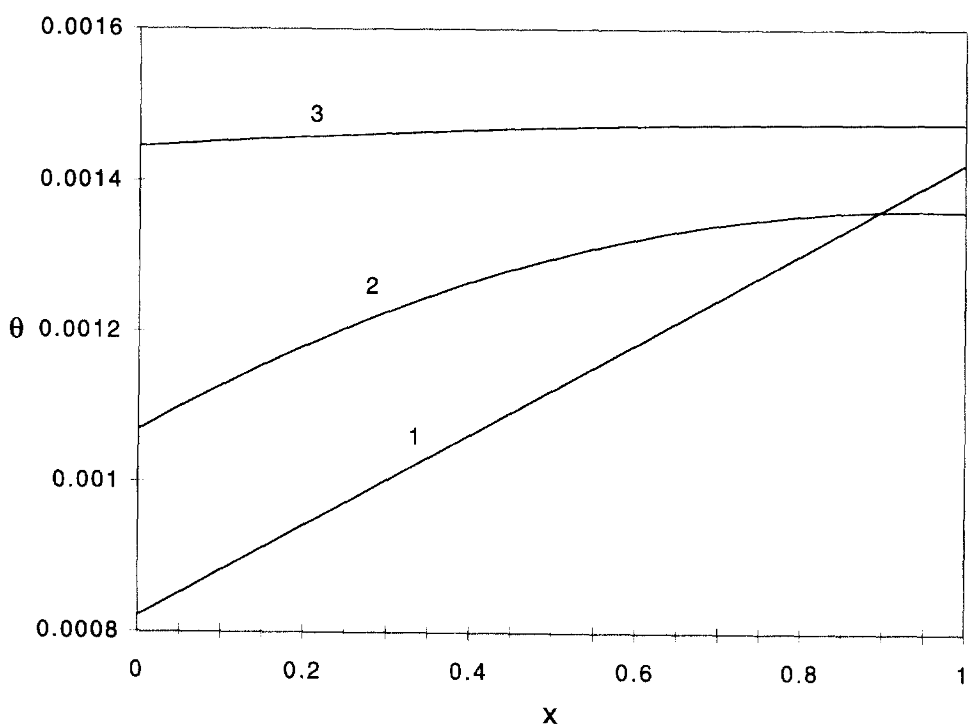

Fig. 4. The temperature is depicted as a function of the axial coordinate x. (1) A uniform conduit, $\alpha = 0.066$. (2) $\alpha = 0.078 - 0.024x^2$. (3) $\alpha = 0.092 - 0.032x - 0.0028x^2$. $\chi = 1.695 \cdot 10^{-6}$.

the axial coordinate x is depicted as curve (2) in Fig. 4. This temperature distribution should be compared with the linear temperature distribution when the conduit's width is uniform. Curve 1 in Fig. 4 depicts the surface temperature as a function of the axial length for a conduit with optimal, uniform width. Witness that the maximum temperature in the case of the non-uniform width channel is about 5% lower than in the case of the uniform, optimal conduit. As functions of the axial coordinate, the widths of the optimal uniform conduit (1) and the non-uniform one (2) are depicted in Fig. 5. Although the maximum surface temperature can be further reduced by increasing the degree of the polynomial (N, equation 8) describing the conduit shape, this issue is not pursued any further since, as noted earlier, the minimum problem does not have a solution. Here, we content ourselves with the demonstration that a non-uniform conduit can lead to a reduction in the maximum temperature.

Figure 4 illustrates yet another potential benefit of using conduits with non-uniform cross-sections. The temperature gradients associated with the curve (2) are much smaller than those associated with the curve (1) of the uniform conduit. This issue can be further exploited by minimizing (7) with $C_2/C_1 = 5 \cdot 10^3$. This particular ratio was chosen so as to make the two terms in equation (7) of similar magnitude. The optimization problem renders

$$\alpha = 0.092 - 0.032x - 0.0028x^2. \tag{11}$$

Witness that the temperature distribution (curve 3 in Fig. 4) is almost horizontal and it can be made more so by increasing the relative magnitude of C_2 in (7). The shape of the conduit which minimizes the temperature gradient is shown as curve 3 in Fig. 5.

5. Conclusions

Using a simple model, it was demonstrated that the width of the uniform conduits of a microheat exchanger can be optimized so as to reduce the maximum temperature of the uniformly heated surface. Additional reductions in the maximal temperature are possible by making the conduit of non-uniform width that varies as a function of the axial coordinate. Non-uniform width conduits can also be used to minimize temperature gradients and render the heated surface temperature nearly uniform. The mode presented here neglects the axial conduction of heat. As a result, the minimization problem for the maximum surface temperature does not have a physically meaningful solution. Here, it was only demonstrated that through the use of a non-uniform conduit, the maximum temperature can be reduced below the best that can be achieved with uniform conduit cross-section. In order to make additional progress, one would need to construct more complicated models than the one pre-

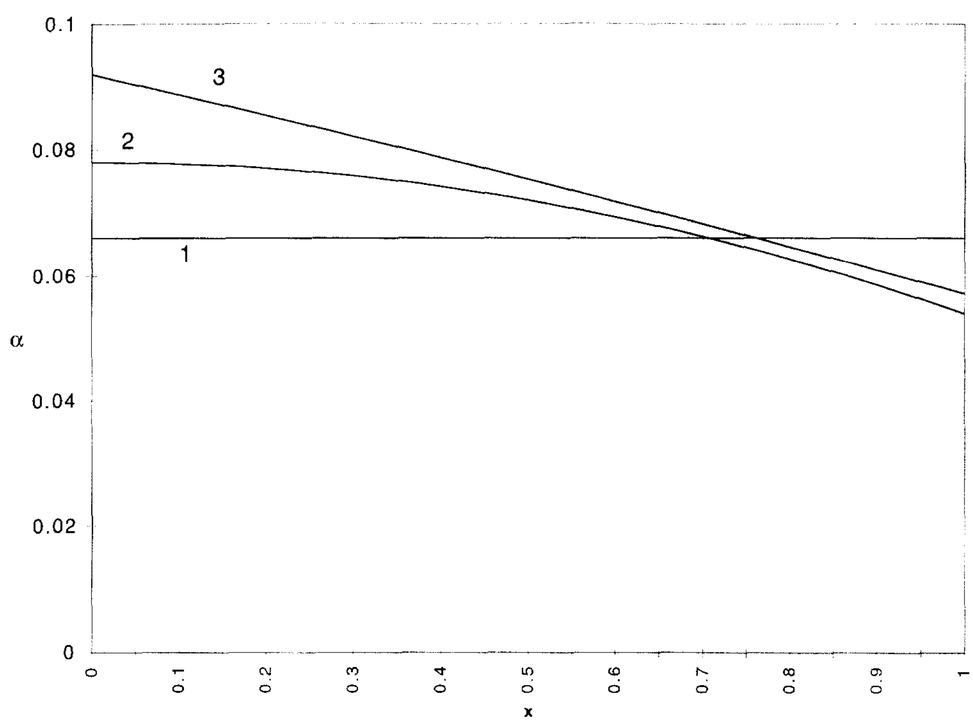

Fig. 5. The optimal conduit shape is depicted as a function of the axial coordinate. (1) A uniform conduit, $\alpha = 0.066$. (2) $\alpha = 0.078 - 0.024x^2$. (3) $\alpha = 0.092 - 0.032x - 0.0028x^2$. $\chi = 1.695 \cdot 10^{-6}$.

sented here which include axial conduction of heat. Such models may require three-dimensional computations of the temperature field. The ideas presented here are not restricted to situations when the heat exchanger's surface is uniformly heated; they also can be applied in cases of non uniformly distributed heating.

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